3033. [2005 : 175, 177] *Proposed by Eckard Specht, Otto-von-Guericke University, Magdeburg, Germany.*

Let I be the incentre of $\triangle ABC$, and let R and r be its circumradius and inradius, respectively. Prove that

$$6r \ \le\ AI+BI+CI \ \le\ \sqrt{12(R^2-Rr+r^2)}\,.$$

I. *Solution by Arkady Alt, San Jose, CA, USA.*

[*Ed:* We give Alt's argument for the left inequality only.]

Let K and s be the area and the semiperimeter of the triangle. Using the well-known (or easy to prove) formulas

$$AI \;=\; \sqrt{\frac{bc(s-a)}{s}}\,,\quad BI \;=\; \sqrt{\frac{ca(s-b)}{s}}\,,\quad CI \;=\; \sqrt{\frac{ab(s-c)}{s}}\,,$$

$abc = 4KR$, $K = sr$, $K = \sqrt{s(s-a)(s-b)(s-c)}$, and the AM–GM Inequality, we obtain

$$\begin{aligned}\frac{AI+BI+CI}{3} \;&\ge\; \sqrt[3]{AI\cdot BI\cdot CI} \;=\; \sqrt[3]{\frac{abc}{s^2}\sqrt{s(s-a)(s-b)(s-c)}}\\ &=\; \sqrt[3]{\frac{abcK}{s^2}} \;=\; \sqrt[3]{\frac{4RK^2}{s^2}} \;=\; \sqrt[3]{4Rr^2}\,.\end{aligned}$$

Thus, $AI+BI+CI \ge 3\sqrt[3]{4Rr^2}$. This inequality is stronger than the one proposed, because Euler's Inequality implies that $3\sqrt[3]{4Rr^2} \ge 6r$.

II. *Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

We give a solution "from the books". The inequality $AI+BI+CI \ge 6r$ is item 12.1 in [1]. On the other hand, item 12.2 in [1] is the inequality $AI+BI+CI \le 2(R+r)$, which is stronger than the proposed one, because the well-known Euler's Inequality $R \ge 2r$ implies that $(R-2r)(2R-r) \ge 0$, and this is equivalent to $2(R+r) \le \sqrt{12(R^2-Rr+r^2)}$.

References

[1] O. Bottema et al., *Geometric Inequalities*, Groningen, 1969

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